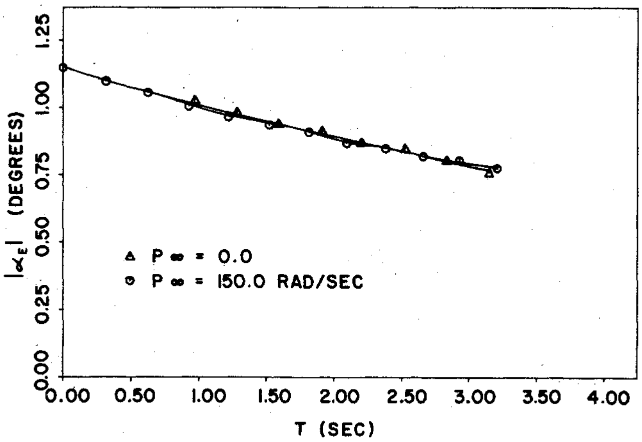

Fig. 1 The effect of P_∞ on $|\alpha_E|$.

Substitute Eq. (10) into Eq. (9) and evaluate r and the a_n's

$$r_j=[M_q(1\pm\tau_\infty)/2I+i(\eta_\infty\pm\omega_0)]/c;\quad j=1,2 \tag{11}$$

and

$$\alpha(t)=A_1\Phi_1(r_1,1+b;RP_1)e^{cr_1t}+A_2\Phi_2(r_2,\ 1-b;RP_1)e^{cr_2t} \tag{12}$$

where

$$R=i(I_x/I)(1/c) \tag{13}$$

$$b=(1/c)[(M_q/I)\tau_\infty+i\omega_0] \tag{14}$$

and,[3]

$$\Phi_{1,2}(r_{1,2},1\pm b;RP_1=1+\sum_{n=1}^{\infty}\frac{(RP_1)^n\prod_{k=1}^{n}(r_{1,2}+k-1)}{n!\prod_{k=1}^{n}(k\pm b)} \tag{15}$$

$A_{1,2}$ are determined by the initial conditions. Observing Eq. (8), (11), (12) and (15) one notes that for

$$M_q<0;\qquad c<0 \tag{16}$$

$$(\dot{K}_j/K_j)<0 \tag{17}$$

for all t.

A similar solution can be found for $p_0\neq 0$.

Numerical Calculations

To check these results, an example that causes Eq. (5) to become positive at $t\approx 0$ was run on a 6-D Computer Program,[4] at the University of Notre Dame. The results are shown in Fig. 1. The data for the computation are presented in Table 1.

Table 1 Data for computations

$C_{M_{p\alpha}}$	$=0$
C_{ℓ_p}	$=-1.76$ 1/rad
C_{M_q}	$=-8.80$ 1/rad
C_{M_α}	$=-0.88$ 1/rad
ω_0	$=10$ rad/sec
V	$=1000$ ft/sec
ρ	$=0.002309$ slugs/ft^3
α_0	$=1.15$ degrees
I_x	$=0.1$ slugs-ft^2
I	$=1.0$ slugs-ft^2
d	$=0.5$ ft
Δt	= integration = 0.001 sec

Conclusions

The final conclusion is that L_p cannot cause λ_j to become positive during flight for any interval of t, and should not, therefore, restrict P_∞ or P_0. However L_p should be considered in the process of evaluating other aerodynamic coefficients from flight data. For this purpose one has to verify that the approximate solution of Eq. (5) is in accordance with the exact solution.

References

[1]Murphy, C. H., "Free Flight Motion of Symmetric Missiles," Ballistic Research Laboratories, Aberdeen Proving Ground, Md., 1216, AD 442757, July 1963.

[2]Nicolaides, J. D., "Free Flight Missile Dynamics," University of Notre Dame, unpublished notes.

[3]Gradshteyn, I. S. and Ryzhik, I. M., *Tables of Integrals, Series and Products*, Academic Press, New York, 1965.

[4]Ingram, C. W., "A Computer Program for Integrating the 6-D of Freedom Equations of Motion of a Symmetrical Missile," University of Notre Dame, unpublished.

Bounds for the Critical Load of Certain Elastic Systems under Follower Forces

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Introduction

ELASTIC systems subjected to follower forces are, in general, nonconservative in inature, and they can have both static and dynamic instabilities, called divergence (buckling) and flutter, respectively.[1] However, there is a class of follower force systems, called the divergence type systems, which can have only divergence instability (see, for example Refs. 2-41). In this Note, an approximate Rayleigh-quotient-type solution for a class of divergence-type systems is presented, and the approximate solution is shown to be either an upper or lower bound to the exact solution, depending on the choice of the approximating deflection function.

Elastic System

Consider an undamped, one-dimensional, linearly elastic system occupying a length ℓ. Let the equation of motion be

$$m(\mathrm{d}^2w/\mathrm{d}t^2)+K(w)+P_cF_c(w)+P_nF_n(w)=0 \tag{1}$$

and the boundary conditions be

$$B(w)=0 \tag{2}$$

where

x = spatial coordinate of the system
t = time
w = predominant deflection of the system from the equilibrium position, $w=w(x,t)$
m = mass density of the system, $m=m(x)$
P_c = conservative component of the external forces
P_n = nonconservative component of the external forces

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K = a self-adjoint operator in x, representing the stiffness of the system

F_c = a self-adjoint operator in x, describing the distribution of the conservative component of the external forces

F_n = a non-self-adjoint operator in x, describing the distribution of the nonconservative components of the external forces

and

B = a self-adjoint operator in x, representing the boundary conditions

The force components P_c and P_n may be related to a single force parameter P by

$$P_c = P\,p_c \tag{3a}$$

$$P_n = P\,p_n \tag{3b}$$

Though Eqs. (1) and (2) describe a nonconservative system, which may or may not be of the divergence type, let us restrict the investigation to divergence-type systems only. Also, let the work done by the nonconservative component of the external forces during any admissible motion be zero, i.e.,

$$\int_0^\ell F_n(\phi)\,dx = 0 \tag{4}$$

where $\phi\ (x)$ is a function which satisfies the boundary conditions (2).

Let the solution of Eq. (1) and (2) be written in the form

$$w(x,t) = e^{i\omega t}y(x) \tag{5}$$

where ω is the frequency of vibration and $i = (-1)^{1/2}$. Substituting Eq. (5) in Eq. (1), we get

$$-m\omega^2 y + K(y) + P_c F_c(y) + P_n F_n(y) = 0 \tag{6}$$

Corresponding Conservative System

When the nonconservative component of the external forces P_n is zero, and the conservative component P_c alone acts on the system, it is called the "corresponding conservative system."[2-4] The equation of motion is given by

$$-m\,\bar{\omega}^2 v + K(v) + P_c F_c(v) = 0 \tag{7}$$

and the boundary conditions are given by

$$B(v) = 0 \tag{8}$$

where $\bar{\omega}$ = frequency of vibration of the corresponding conservative system, and $v = v(x)$ = predominant deflection function of the corresponding conservative system. Let the first eigenmode of Eq. (7) and (8), at $\bar{\omega}^2 = 0$ be $v_1(x)$. This is called the fundamental buckling mode of the corresponding conservative system.

Corresponding Free System

When both of the conservative and nonconservative components of the external forces P_c and P_n are zero, the system is called the "corresponding free system." The equation of motion is

$$-m\Omega^2 u + K(u) = 0 \tag{9}$$

and the boundary conditions are

$$B(u) = 0 \tag{10}$$

where Ω = natural frequency of the system, and $u = u(x)$ = predominant deflection function. Let the first eigenmode of Eqs. (9) and (10) be $u_1(x)$. This is called the fundamental vibration mode of the corresponding free system.

Approximate Solutions

Let us obtain an approximate solution to the corresponding conservative system, Eq. (7) and (8), by approximating the deflection function by the fundamental vibration mode of the corresponding free system, i.e.,

$$v(x) = u_1(x) \tag{11}$$

The Rayleigh quotient is

$$\bar{\omega}^2{}_{,u} = \frac{\int_0^\ell K(u_1)u_1 dx + P_c \int_0^\ell F_c(u_1)u_1 dx}{\int_0^\ell m\,u_1^2 dx} \tag{12}$$

where $\bar{\omega}_{,u}$ is an approximation to $\bar{\omega}$. This is a straight line in the load-frequency square plane, and may be called the "Rayleigh line" (Fig. 1).

A similar approximate solution to the nonconservative problem, Eq. (6), is

$$\omega^2{}_{,u} = \frac{\int_0^\ell K(u_1)u_1 dx + P_c \int_0^\ell F_c(u_1)u_1 dx + P_n \int_0^\ell F_n(u_1)u_1 dx}{\int_0^\ell m\,u_1^2 dx} \tag{13}$$

where $\omega_{,u}$ is an approximation to ω. The last term of the numerator is zero, because of Eq. (4), and hence the left-hand side of Eqs. (12) and (13) are identical. So, both the nonconservative system and the corresponding conservative system have the same Rayleigh line.

The Rayleigh line passes through the point $\omega_0^2 = \bar{\omega}_0^2 = \Omega 2$, where ω_0 and $\bar{\omega}_0$ are the values of ω and $\bar{\omega}$, respectively, at $P=0$; also, the Rayleigh line is an upperbound to the first branch of the eigencurve† (load vs frequency square curve) of the corresponding conservative system. Huseyin and Leipholz[3]‡ have shown that the fundamental eigencurves of the divergence-type nonconservative system and the corresponding conservative system are tangential to each other at $\omega_0^2 = \bar{\omega}_0^2 = \Omega^2$, and that the fundamental eigencurve of the divergence-type system is not convex towards the origin. From the above three conditions, we conclude that the Rayleigh line is also an upper bound for the fundamental eigencurve of the nonconservative system under consideration.

An approximate critical load for the nonconservative system can be obtained by letting $\omega^2{}_{,u} = 0$ in Eq. (13):

$$P_{cr,u} = \left[-\int_0^\ell K(u_1)u_1 dx\right] \Big/ \left[p_c \int_0^\ell F_c(u_1)u_1 dx\right] \tag{14}$$

where $P_{cr,u}$ is an approximation to the critical load P_{cr}. Since the Rayleigh line is an upper bound to the eigencurve,

$$P_{cr} \leq P_{cr,u} \tag{15}$$

†Henceforth, the first branch of the eigencurve will be called the fundamental eigencurve.

‡Reference 3 deals with a discrete N-degree-of-freedom system, valid for any positive integer value of N. Since a continuous system can be approximated to an N-degree-of-freedom system by Galerkin's method, and the absolute and uniform convergence of the Galerkin's method has been established for a wide class of one-dimensional nonconservative systems,[5-7] the results of Ref. 3 are valid for the present one-dimensional system under investigation.

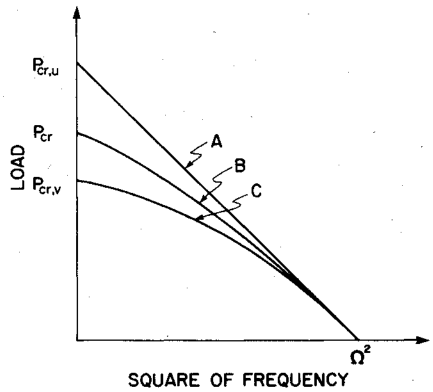

Fig. 1 Eigencurves: A) Rayleigh line; B) eigencurve of the nonconservative system; C) eigencurve of the corresponding conservative system.

The exact critical load of the corresponding conservative system is

$$\bar{P}_{cr} = \frac{-\int_0^\ell K(v_1)v_1 \mathrm{d}x}{p_c \int_0^\ell F_c(v_1)v_1 \mathrm{d}x} \tag{16}$$

Let us now obtain an approximate critical load for the nonconservative system, using the fundamental buckling mode of the corresponding conservative system as the approximate deflection function, i.e.,

$$y(x) = v_1(x) \tag{17}$$

Using Galerkin's method, we get:

$$P_{cr,v} = \frac{-\int_0^\ell K(v_1)v_1 \mathrm{d}x}{p_c \int_0^\ell F_c(v_1)v_1 \mathrm{d}x + p_n \int_0^\ell F_n(v_1)v_1 \mathrm{d}x} \tag{18}$$

The second term in the denominator is zero, because of Eq. (4). Comparison of Eqs. (16) and (18) yields,

$$P_{cr,v} = \bar{P}_{cr} \tag{19}$$

Leipholz and Huseyin[3] have shown that the critical load of the corresponding conservative system is a lower bound to the critical load of the preceding divergence-type nonconservative system, i.e.,

$$P_{cr} \geq \bar{P}_{cr} \tag{20}$$

From Eqs. (19) and (20),

$$P_{cr} \geq P_{cr,v} \tag{21}$$

Equations (12) and (21) may be summarized by the following Theorem: the approximate critical load of an undamped, one-dimensional, divergence-type nonconservative elastic system, in which the work done by the nonconservative component of the external forces is zero, is an upper bound, if the deflection of the system is approximated by the fundamental vibration mode of the corresponding free system; it is a lower bound if the deflection of the system is approximated by the fundamental buckling mode of the corresponding conservative system.

References

[1]Bolotin, V. V., *Nonconservative Problems of the Theory of Elastic Stability*, Pergamon Press, New York, 1963.

[2]Leipholz, H. H. E. and Huseyin, K., "On the Stability of One-Dimensional Continuous Systems with Polygenic Forces," *Meccanica*, Vol. 6, 1971, pp. 253-257.

[3]Huseyin, K. and Leipholz, H. H. E., "Divergence Instability of Multiple-Parameter Circulatory Systems," *Quarterly of Applied Mathematics*, Vol. 31, 1973, pp. 185-197.

[4]Leipholz, H. H. E., "On Certain Nonconservative Elastic Systems Having Divergence Buckling Loads," *Mechanics Research Communications*, Vol. 1, 1974, pp. 245-249.

[5]Leipholz, H. H. E., "Anwendung des Galerkinschen Verfahrens auf nichtkonservative Stabilitats probleme des elastichen Stabes," *Zeitschrift fur Angewandte Mathematik und Physik*, Vol. 13, 1962, pp. 359-372.

[6]Leipholz, H. H. E., "Uber die Konvergenz des Galerkinschen Verfahrens bei nichtselbstadjungierten und nichtkonservativen Eigenvert problemen," *Zeitsehrift fur Angewandte Mathematik und Physik*, Vol. 14, 1963, pp. 70-79.

[7]Leipholz, H. H. E., "Uber die Zulassigkeit des Verfahren von Galerkin bei linearen, nichtselbstadjungierten Eigenvert problemen," *Zeitschrift fur Angewandte Mathematik und Physik*, Vol. 16, 1965, pp. 837-843.

Experimental Investigation of Underexpanded Exhaust Plumes

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IN a recent paper,[1] one of the authors presented an approximate model for calculating highly underexpanded rocket exhaust plumes. The aim of the present work is to compare original experimental results with the predictions of the reported model.

Calculations

Our calculations are based on the following assumptions:

1) The flowfield is divided into two regions separated by a thin viscous layer: the external atmosphere and an inner region, where the exhaust gas is unaffected by the external atmosphere, and thus behaves as in a vacuum expansion.[2]

2) Within the inner region, the streamlines are radial, the local velocity U is equal to its limiting value U_L, and the evolution is isentropic along the streamlines. The exhaust gas is assumed to be perfect with constant specific heat ratio γ. Considering polar coordinates R, θ centered on the source, the continuity equation yields

$$(\rho U_L)/(\rho_e U_e) = A\, f(\theta)/R^2, \text{ with } f(\theta=0)=1 \tag{1}$$

where ρ is the density, A is a constant, and the subscript e refers to the nozzle exit conditions.

3) The inner region is divided into two parts[3]: for small polar angles ($\theta<\theta_o$), the gas originates from the isentropic core of the nozzle, whereas for large polar angles ($\theta>\theta_0$), the gas originates from the nozzle boundary layer. Semiempirical expressions for $f(\theta)$ in both regions have been proposed by Boynton[4] and Simons.[3] Details of the model may be found in Ref. 5. It essentially consists of the model described in Ref. 1, except for some improvements which allow calculations with thick nozzle boundary layers to be made. Thus, the equation

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